Hydromagnetic Stability of Two Rivlin-Ericksen Elastico-Viscous Superposed Conducting Fluids

R. C. Sharma and P. Kumar

Department of Mathematics, Himachal Pradesh University, Summer Hill, Shimla-171 005, India

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The stability of the plane interface separating two Rivlin-Ericksen elastico-viscous superposed fluids of uniform densities when the whole system is immersed in a uniform horizontal magnetic field has been studied. The stability analysis has been carried out, for mathematical simplicity, for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that the stability criterion is independent of the effects of viscosity and viscoelasticity and is dependent on the orientation and magnitude of the magnetic field. The magnetic field is found to stabilize a certain wave-number range of the unstable configuration. The behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity parameters are examined numerically.

1. Introduction

The instability of the plane interface separating two Newtonian fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors, and Chandrasekhar [1] has given a detailed account of these investigations. Roberts [2] has extended the analysis to the case of two fluids of equal kinematic viscosities in presence of a vertical magnetic field, while Gerwin [3] has studied the case of compressible streaming fluids. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field has been studied by Bhatia [4]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities. A good account of hydrodynamic stability problems has also been given by Drazin and Reid [5] and Joseph [6].

The fluids have been considered to be Newtonian in all the above studies. The stability of a layer of viscoelastic (Oldroyd) fluid heated from below and subject to a magnetic field has been studied by Sharma [7]. In another study Sharma and Sharma [8] have studied the stability of the plane interface separating two viscoelastic (Oldroyd) superposed fluids of uniform densities. Fredricksen [9] has given a good review of non-Newtonian fluids whereas Joseph [6] has also considered the stability of viscoelastic fluids. Molten plastics, petroleum oil additives and whipped cream are examples of incompressible viscoelastic fluids. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [10] constitutive relations. The Rivlin-Ericksen elastico-viscous fluid is one such fluid. It is this class of elastico-viscous fluids we are interested in particularly to study the stability of the plane interface separating two incrompressible superposed Rivlin-Ericksen fluids of uniform densities pervaded by a uniform horizontal magnetic field in addition to a constant gravity field. This aspect forms the subject of the present paper where we have carried out the stability analysis for two fluids of equal kinematic viscosities, equal kinematic viscoelasticities and different densities.

2. Perturbation Equations

Consider a static state, in which an incompressible, infinitely conducting, Rivlin-Ericksen elastico-viscous fluid of variable density pervaded by a uniform magnetic field $\boldsymbol{H}(H_x, H_y, 0)$ is arranged in horizontal strata and the pressure p and density ϱ are functions of the vertical coordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\boldsymbol{q}(u, v, w)$, $\delta\varrho$, δp and $\boldsymbol{h}(h_x, h_y, h_z)$ denote the perturbations in velocity (0, 0, 0), density ϱ, pressure p and magnetic field $\boldsymbol{H}$, respectively. Then the linearized hydromagnetic perturbation equations relevant to the

Reprint requests to Prof. R. C. Sharma.

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problem are

$$\varrho \frac{\partial \boldsymbol{q}}{\partial t} = -\nabla \delta p + \boldsymbol{g}\,\delta\varrho + \varrho\left(v + v'\frac{\partial}{\partial t}\right)\nabla^2 \boldsymbol{q} + \frac{1}{4\pi}(\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + \left(\frac{\mathrm{d}\mu}{\mathrm{d}z} + \frac{\partial}{\partial t}\frac{\mathrm{d}\mu'}{\mathrm{d}z}\right)\left(\frac{\partial w}{\partial \bar{x}} + \frac{\partial \boldsymbol{q}}{\partial z}\right), \quad (1)$$

$$\nabla \cdot \boldsymbol{q} = 0, \quad (2)$$

$$\nabla \cdot \boldsymbol{h} = 0, \quad (3)$$

$$\frac{\partial}{\partial t}\delta\varrho = -w\,D\varrho, \quad (4)$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla \times (\boldsymbol{q} \times \boldsymbol{H}), \quad (5)$$

where $v\left(=\frac{\mu}{\varrho}\right)$ and $v'\left(=\frac{\mu'}{\varrho}\right)$ denote the kinematic viscosity and kinematic viscoelasticity of the fluid, $\boldsymbol{g} = (0, 0, -g)$ is the acceleration due to gravity, $\bar{x} = (x, y, z)$, and $D = \frac{\mathrm{d}}{\mathrm{d}z}$. Since the density of a fluid particle remains unchanged if we follow it with its motion, we have

$$\frac{\partial \varrho}{\partial t} + (\boldsymbol{q} \cdot \nabla)\varrho = 0.$$

This additional equation must be satisfied since the fluid is heterogeneous. For a homogeneous fluid, this is identically satisfied and (4) is its linearized perturbed form.

Decomposing disturbances into eigen modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(i k_x x + i k_y y + n t), \quad (6)$$

where k_x, k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$, and n is a complex constant.

For perturbations of the form (6), (1)–(5) give

$$\varrho n u = -i k_x \delta p + \varrho(v + v' n)(D^2 - k^2) u + \frac{H_y}{4\pi}(i k_y h_x - i k_x h_y) + (D\mu + n D\mu')(i k_x w + D u), \quad (7)$$

$$\varrho n v = -i k_y \delta p + \varrho(v + v' n)(D^2 - k^2) v + \frac{H_x}{4\pi}(i k_x h_y - i k_y h_x) + (D\mu + n D\mu')(i k_x w + D v), \quad (8)$$

$$\varrho n w = -D\delta p - g\,\delta\varrho + \varrho(v + v' n)(D^2 - k^2) w + \frac{H_x}{4\pi}(i k_x h_z - D h_x) + \frac{H_y}{4\pi}(i k_y h_z - D h_y) + 2(D\mu + n D\mu') D w, \quad (9)$$

$$i k_x u + i k_y v + D w = 0, \quad (10)$$

$$i k_x h_x + i k_y h_y + D h_z = 0, \quad (11)$$

$$n\,\delta\varrho = -w\,D\varrho, \quad (12)$$

$$n \boldsymbol{h} = (i k_x H_x + i k_y H_y)\boldsymbol{q}. \quad (13)$$

Eliminating u, v, h_x, h_y, h_z, $\delta\varrho$ and δp in (7)–(9) with the help of (10)–(13), we obtain

$$\begin{aligned} & n[D(\varrho D w) - k^2 \varrho w] + \frac{g k^2}{n}(D\varrho) w \\ & - [D\{\varrho(v + v' n)(D^2 - k^2) D w\} \\ & \quad - k^2 \varrho (v + v' n)(D^2 - k^2) w] \\ & + \frac{1}{4\pi n}(k_x H_x + k_y H_y)^2 (D^2 - k^2) w \\ & - [D\{(D\mu + n D\mu')(D^2 + k^2) w\} \\ & \quad - 2k^2 (D\mu + n D\mu') D w] = 0. \end{aligned} \quad (14)$$

3. Two Superposed Rivlin-Ericksen Fluids Separated by a Horizontal Boundary

Here we consider the case when two Rivlin-Ericksen superposed fluids of uniform densities ϱ_1, ϱ_2, viscosities μ_1, μ_2 and viscoelasticities μ_1', μ_2' are separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 indicate the lower and upper fluid. Then, in each region of constant ϱ, constant μ and constant μ', (14) reduces to

$$(D^2 - k^2)(D^2 - \kappa^2) w = 0, \quad (15)$$

where

$$\kappa^2 = k^2 + \frac{n}{v + v' n}\left[1 + \frac{1}{4\pi n^2 \varrho}(k_x H_x + k_y H_y)^2\right]. \quad (16)$$

Since w must vanish both when $z \to +\infty$ (in the upper fluid) and $z \to -\infty$ (in the lower fluid), the general solution of (15) can be written as

$$w_1 = A_1 e^{+kz} + B_1 e^{+\kappa_1 z} \; (z < 0), \quad (17)$$

$$w_2 = A_2 e^{-kz} + B_2 e^{-\kappa_2 z} \; (z > 0), \quad (18)$$

where A_1, B_1, A_2, B_2 are constants of integration,

$$\kappa_1 = \left[k^2 + \frac{n}{v_1 + n v_1'}\left\{1 + \frac{(k_x H_x + k_y H_y)^2}{4\pi \varrho_1 n^2}\right\}\right]^{1/2}, \quad (19)$$

and

$$\kappa_2 = \left[k^2 + \frac{n}{v_2 + n v_2'}\left\{1 + \frac{(k_x H_x + k_y H_y)^2}{4\pi \varrho_2 n^2}\right\}\right]^{1/2}. \quad (20)$$

In writing (19) and (20), it is assumed that κ_1 and κ_2 are so defined that their real parts are positive.

4. Boundary Conditions

The solutions (17) and (18) must satisfy certain boundary conditions. Clearly, all three components of velocity and tangential viscous stresses must be continuous. The continuity of Dw follows from (10) and the continuity of u and v. Since

$$\tau_{xz} = \left(2\mu + 2\mu' \frac{\partial}{\partial t}\right) e_{xz} = (\mu + \mu' n)(Du + i k_x w)$$

and

$$\tau_{yz} = \left(2\mu + 2\mu' \frac{\partial}{\partial t}\right) e_{yz} = (\mu + \mu' n)(Dv + i k_y w)$$

are continuous,

$$i k_x \tau_{xz} + i k_y \tau_{yz} = -(\mu + \mu' n)(D^2 + k^2) w$$

is continuous across an interface between the two fluids. Hence the boundary conditions to be satisfied at the interface $z = 0$ are that

$$w\,, \tag{21}$$

$$Dw \tag{22}$$

and

$$(\mu + \mu' n)(D^2 + k^2) w \tag{23}$$

must be continuous.

Integrating (14) across the interface $z = 0$, we obtain another condition:

$$\begin{aligned}
&[\varrho_2 D w_2 - \varrho_1 D w_1]_{z=0} - \left[\frac{1}{n}(\mu_2 + n\mu_2')(D^2 - k^2) D w_2 - \frac{1}{n}(\mu_1 + n\mu_1')(D^2 - k^2) D w_1\right]_{z=0} \\
&+ \frac{(k_x H_x + k_y H_y)^2}{4\pi n^2}(D w_2 - D w_1)_{z=0} \\
&= -\frac{g k^2}{n^2}[\varrho_2 - \varrho_1] w_0 \\
&- \frac{2k^2}{n}(\mu_2 + n\mu_2' - \mu_1 - n\mu_1')(Dw)_0\,,
\end{aligned} \tag{24}$$

where w_0 and $(Dw)_0$ are the common values of w_1, w_2 and Dw_1, Dw_2, respectively, at $z = 0$.

5. Dispersion Relation and Discussion

Applying the boundary conditions (21)–(24) to the solutions (17) and (18), we obtain

$$\det(a_{ij}) = 0\,, \tag{25}$$

where $i, j = 1, 2, 3, 4$ and

$$\begin{aligned}
&a_{11} = a_{12} = 1\,, \quad a_{13} = a_{14} = -1\,, \quad a_{21} = a_{23} = k\,, \quad a_{22} = \kappa_1\,, \quad a_{24} = \kappa_2\,, \quad a_{31} = 2k^2(\mu_1 + n\mu_1')\,, \\
&a_{32} = (\mu_1 + n\mu_1')(\kappa_1^2 + k^2)\,, \quad a_{33} = -2k^2(\mu_2 + n\mu_2')\,, \quad a_{34} = -(\mu_2 + n\mu_2')(\kappa_2^2 + k^2)\,, \\
&a_{41} = \left[-\alpha_1 + \frac{R}{2} + \frac{k^2}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) - \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right], \\
&a_{42} = \left[\frac{R}{2} + \frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)\kappa_1\right], \\
&a_{43} = \left[-\alpha_2 + \frac{R}{2} - \frac{k^2}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) - \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right], \\
&a_{44} = \left[\frac{R}{2} - \frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)\kappa_2\right], \\
&\alpha_1 = \frac{\varrho_1}{\varrho_1 + \varrho_2}\,, \quad \alpha_2 = \frac{\varrho_2}{\varrho_1 + \varrho_2}\,, \quad R = \frac{gk}{n^2}(\alpha_2 - \alpha_1)\,, \quad (\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 = \frac{(k_x H_x + k_y H_y)^2}{4\pi(\varrho_1 + \varrho_2)}\,.
\end{aligned} \tag{26}$$

Here $\boldsymbol{k}$ is the wave number vector and $\boldsymbol{V}_A$ is the Alfvén velocity vector.

Equation (25) yields the following characteristic equation

$$
\begin{aligned}
&(\kappa_1 - k)\Big[-2k^2(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \\
&\quad \cdot \left\{-\frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)(\kappa_2 - k) + \alpha_2 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\} \\
&\quad + \left\{R - 1 - \frac{2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\}\{(\nu_2\alpha_2 + n\nu_2'\alpha_2)(\kappa_2^2 - k^2)\}\Big] \\
&- 2k\Big[(\nu_1\alpha_1 + n\nu_1'\alpha_1)(\kappa_1^2 - k^2)\left\{-\frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)(\kappa_2 - k) + \alpha_2 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\} \\
&\quad + (\nu_2\alpha_2 + n\nu_2'\alpha_2)(\kappa_2^2 - k^2)\left\{\frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)(\kappa_1 - k) + \alpha_1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\}\Big] \\
&+ (\kappa_2 - k)\Big[(\nu_1\alpha_1 + n\nu_1'\alpha_1)(\kappa_1^2 - k^2)\left\{R - 1 - \frac{2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\} + 2k^2(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \\
&\quad \cdot \left\{\frac{k}{n}(\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)(\kappa_1 - k) + \alpha_1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\}\Big] = 0 . \qquad (27)
\end{aligned}
$$

The dispersion relation (27) is quite complicated, as the values of κ_1 and κ_2 involve square roots. We therefore make the assumption that the two fluids are of high viscosity and high viscoelasticity. Under this assumption, we have

$$
\begin{aligned}
\kappa &= k\left[1 + \frac{n}{(\nu + n\nu')k^2}\left\{1 + \frac{(k_x H_x + k_y H_y)^2}{4\pi\varrho n^2}\right\}\right]^{1/2} \\
&= k + \frac{n}{2(\nu + n\nu')k}\left\{1 + \frac{(k_x H_x + k_y H_y)^2}{4\pi\varrho n^2}\right\},
\end{aligned}
$$

so that

$$
\kappa_1 - k = \frac{n}{2(\nu_1 + n\nu_1')k}\left\{1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{\alpha_1 n^2}\right\} \qquad (28)
$$

and

$$
\kappa_2 - k = \frac{n}{2(\nu_2 + n\nu_2')k}\left\{1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{\alpha_2 n^2}\right\}. \qquad (29)
$$

Substituting the values of $\kappa_1 - k$ and $\kappa_2 - k$ from (28) and (29) in (27) and putting $\nu_1 = \nu_2 = \nu$, $\nu_1' = \nu_2' = \nu'$ (the case of equal kinematic viscosities and equal kinematic viscoelasticities for mathematical simplicity, but this assumption does not affect the stability analysis qualitatively), we obtain the dispersion relation

$$
\begin{aligned}
&[\alpha_1\alpha_2(1 + 2k^2\nu')]n^6 + [2k^2\alpha_1\alpha_2\nu]n^5 \\
&+ [\alpha_1\alpha_2\{2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 - gk(\alpha_2 - \alpha_1)\} \\
&+ (\boldsymbol{k}\cdot\boldsymbol{V}_A)^2(1 + 2k^2\nu')]n^4 + [2k^2\nu(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2]n^3 \\
&+ [(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2\{2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 - gk(\alpha_2 - \alpha_1)\} \\
&+ (\boldsymbol{k}\cdot\boldsymbol{V}_A)^4(1 + 2k^2\nu')]n^2 + [2k^2\nu(\boldsymbol{k}\cdot\boldsymbol{V}_A)^4]n \\
&+ [(\boldsymbol{k}\cdot\boldsymbol{V}_A)^4\{2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 - gk(\alpha_2 - \alpha_1)\}] = 0 . \qquad (30)
\end{aligned}
$$

For the potentially stable arrangement $\alpha_1 > \alpha_2$, (30) does not involve any change of sign and so does not allow any positive root. The system is therefore stable. This result is also true when both the fluids are viscous (Chandrasekhar [1]) or Oldroydian viscoelastic (Sharma [11]).

For the potentially unstable configuration $\alpha_2 > \alpha_1$, if

$$
2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 > gk(\alpha_2 - \alpha_1), \qquad (31)
$$

(30) does not admit any change of sign and so has no positive root. Therefore the system is stable.

However, if

$$
2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 < gk(\alpha_2 - \alpha_1), \qquad (32)
$$

the constant term in (30) is negative. Equation (30), therefore, allows one change of sign and so has one positive root. The occurrence of a positive root implies that the system is unstable.

Thus for the unstable case $\alpha_2 > \alpha_1$, the magnetic field has got a stabilizing effect and the system is stable for all wave numbers which satisfy the inequality

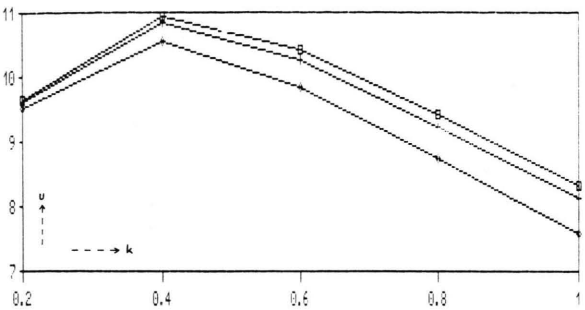
$$
2(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2 > gk(\alpha_2 - \alpha_1),
$$

i.e.

$$
2k(V_1\cos\Theta + V_2\sin\Theta)^2 > g(\alpha_2 - \alpha_1), \qquad (33)
$$

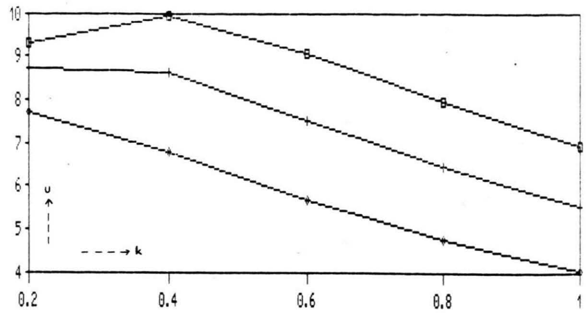
where V_1 and V_2 are the Alfvén velocities in x and y directions and Θ is the angle between $\boldsymbol{k}$ and H_x.

The stability criterion (31) is independent of the effect of viscosity and viscoelasticity. The magnetic



□: $\nu = 1$, +: $\nu = 2$, ◇: $\nu = 5$

Fig. 1. The variation of the growth rate c (positive real value of n) with the wave number k for the kinematic viscosities $\nu = 1, 2, 5$ when $\alpha_1 = 0.38$, $\alpha_2 = 0.62$, $\nu' = 2$, $V_A = 15$ and $\Theta = 45°$.



□: $\nu' = 3$, +: $\nu' = 5$, ◇: $\nu' = 10$

Fig. 2. The variation of the growth rate c (positive real value of n) with the wave number k for the kinematic visco-elasticities $\nu' = 3, 5, 10$ when $\alpha_1 = 0.38$, $\alpha_2 = 0.62$, $\nu = 2$, $V_A = 15$ and $\Theta = 45°$.

field stabilizes a certain wave-number range of the unstable configuration even in the presence of the effects of viscoelasticity. The critical wave number k^*, above which the system is stabilized, depends on the magnitudes V_1 and V_2 of the magnetic field as well as the orientation of the magnetic field Θ.

We now examine the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity numerically. We have plotted the growth rate c (positive real value of n) versus the wave number k for several values of the kinematic viscosity ν and the kinematic viscoelasticity ν' in Figs. 1 and 2, respectively.

It is seen from Fig. 1 that for the same wave number k the growth rate c decreases as the kinematic viscosity ν increases, and that for the same kinematic viscosity ν the growth rate increases for low wave numbers and decreases for high wave numbers. Similar effects can be seen from Fig. 2 for the kinematic viscoelasticity.

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